DYNAMIC PROGRAMMING AND THE COMPUTATIONAL SOLUTION OF FEEDBACK DESIGN CONTROL PROBLEMS

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SUMMARY

In this paper we indicate how a class of control processes requiring multi-dimensional sequences of functions when treated by the direct methods of dynamic programming can, by means of a transformation familiar to the theory of linear functional equations of differential, difference, or differential-difference type, be reduced to problems involving sequences of functions of one variable in a number of cases, and sequences of functions of two variables in others.

These results open the door to a systematic study of non-linear control processes, with or without time-lags and other types of hereditary behavior, by way of the method of successive approximations.

As an example, we consider the problem of minimizing

$$J(v) = |u(10)| + \lambda \int_0^{10} v^2 dt$$

over all functions $v(t)$ satisfying the constraint $-.1 \leq v(t) \leq 0$, $0 \leq t \leq 10$, where $u'' + .3u' + .02u = 0$, $u(0) = 2$, $u'(0) = -.3$.

CONTENTS

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1. INTRODUCTION

In this paper we wish to provide an expository account of some new dynamic programming techniques we have recently developed for the express purpose of handling variational problems arising in the design of feedback control circuits.

Assuming that the reader is familiar with the physical background of these questions, we shall concern ourselves exclusively with some of the mathematical problems involved. Detailed accounts of these processes may be found in the recent books by Laning and Battin, [16], and by Truxal, [17]. Interesting accounts of other analytic techniques used to study problems in the theory of control may be found in Bass, [1], Booton, [14], and Kalman, [15].

Our aim is to provide a straightforward method for obtaining the numerical solution of various classes of control problems, based upon the use of a modern digital computer. Although a certain amount of analytic preparation is required, necessitating the fundamentals of the theory of linear differential or difference equations, our fundamental purpose is to furnish a computational technique which bypasses any detailed analytic study of the equations governing the process under consideration.

In order to keep the paper as self-contained as possible,

we shall present a brief exposition of the functional equation technique of dynamic programming. A more detailed account of this theory may be found in our book, [2], and the accounts in [3] and [4], may also be helpful. Further mathematical discussion is contained in [5], [6], [7].

We have included a simple example of the application of these techniques, in order to illustrate the time and effort required in handling these problems.

2. A GENERAL FEEDBACK CONTROL PROBLEM

Let us begin by formulating a general control problem, restricting ourselves, however, for the sake of notational simplicity, to systems governed by second order differential equations.

Given an equation of the form

$$(1) \qquad \frac{d^2u}{dt^2} = g\left(\frac{du}{dt}, u\right) + v(t), \quad u(0) = c_1, \quad u'(0) = c_2,$$

where $v(t)$ is a forcing function or control function, resulting from feedback, we wish to determine $v(t)$ in such a way as to minimize an expression of the form

$$(2) \qquad J(v) = \int_0^T h\left(u, \frac{du}{dt}\right)dt + \int_0^T q(v)dt + F(x(T), x'(T)).$$

The function v will usually be subject to additional constraints of the form

$$(3) \quad \begin{aligned} &\text{a.} \quad m \leq v(t) \leq m', \quad 0 \leq t \leq T, \\ &\text{b.} \quad \int_0^T r(v)dt \leq b. \end{aligned}$$

In a number of situations, these terms have the following interpretation. The first integral, $\int_0^T h(u,du/dt)dt$, represents the cost incurred due to the deviation of the system from some desired operating state over the time period $[0,T]$; the second integral, $\int_0^T q(v)dt$, represents the cost of using control over that time period; the third term represents an additional cost due to the final state of the system at the end of the process.

If the first term is absent, we speak of a <u>terminal control</u> process. Actually, there is no clear-cut distinction, since various transformations will convert one type of process into another.

The restrictions on v are of physical nature, corresponding to constraints concerning the maximum voltage or current that can be used, or in physical control processes, to a maximum angle, maximum rate of turn, and so on. Finally, the last constraint corresponds to an energy constraint on the system, or in economic terms, a constraint on the quantity of resources available for control.

The discussion in [4] shows that there is a complete analogy between feedback control problems for engineering and economic control processes, and the same mathematical techniques can be used in each domain.

Problems of the foregoing type are, in general, extremely difficult to solve analytically in traditional fashion, i.e. in terms of explicit solutions of differential equations, due not only to the nonlinearity of the equations but also, even in linear cases, to the presence of constraints. A discussion of various classes of problems of this type, together with some explicit solutions, may be found in [12] and [13].

3. DYNAMIC PROGRAMMING APPROACH

Let us now briefly sketch the application of the theory of dynamic programming to the computational solution of problems of the above type. Introduce the function of three variables $f(c_1,c_2,T)$ by means of the relation

$$(1) \qquad f(c_1,c_2,T) = \operatorname*{Min}_{v} J(u),$$

where $J(u)$ is as in (2.2) and, for the moment the function satisfies only the constraint (2.3a).

The quantities c_1 and c_2 are the state variables specifying the initial condition of the system and T is the duration of the process.

Under various conditions, which are usually met in applications, we obtain for f the nonlinear partial differential equation

$$(2) \qquad \frac{\partial f}{\partial T} = \operatorname*{Min}_{v} \left[h(c_1,c_2) + q(v) + c_2 \frac{\partial f}{\partial c_1} + [v + g(c_1,c_2)] \frac{\partial f}{\partial c_1} \right],$$

where the quantity v ranges over the interval $[m,m']$. The

initial condition is

(3) $$f(c_1,c_2,0) = 0.$$

For the details of this, see [2].

This equation can be treated computationally by means of the usual approximation to the partial derivatives by partial differences. It has turned out in practice, however, that it is better to proceed as follows. In place of the continuous process described by the equation in (2.1), which we write in the form

(4) $$\frac{du}{dt} = w, \quad u(0) = c_1, \qquad \frac{dw}{dt} = g(u,w) + v, \quad w(0) = c_2,$$

(a phase space representation), we consider the discrete process described by the difference equations

(5) $$u_{k+1} - u_k = w_k\delta, \quad u_0 = c_1, \qquad w_{k+1} - w_k = [g(u_k,w_k) + v_k]\delta, \quad w_0 = c_2,$$

$k = 0, 1, 2, \ldots, N-1$.

Here

(6) $$u_k = u(k\delta), \quad v_k = v(k\delta), \quad w_k = w(k\delta), \quad k = 0,1,2,\ldots,N-1,$$

and $N\delta = T$.

In place of minimizing the functional $J(u)$, we minimize

the function of N variables

$$(7) \qquad J(v_0, v_1, \ldots, v_{N-1}) = \left[\delta \sum_{k=0}^{N-1} h(u_k, w_k) + \delta \sum_{k=0}^{N-1} q(v_k) \right].$$

The principle of optimality, cf. [2], yields in place of (2) the relation

$$(8) \qquad \begin{aligned} f_K(c_1, c_2) = \operatorname*{Min}_{v} \Big[&\delta\, h(c_1, c_2) + \delta\, q(v_0) \\ &+ f_{K-1}\left(c_1 + \delta c_2, c_2 + \delta [g(c_1, c_2) + v_0]\right) \Big] \end{aligned}$$

where the sequence $\{f_K(c_1, c_2)\}$ is defined by

$$(9) \qquad f_K(c_1, c_2) = \operatorname*{Min}_{v} J(v_0, v_1, \ldots, v_{K-1}), \quad K \geq 1.$$

4. LAGRANGE MULTIPLIER

In order to take account of the constraint of (2.3b) without adding another state variable, we introduce a Lagrange multiplier, cf. [8], and consider the new function

$$(1) \qquad \begin{aligned} J(v_0, v_1, \ldots, v_{N-1}) = \Big[&\delta \sum_{k=0}^{N-1} h(u_k, w_k) \\ &+ \delta \sum_{k=0}^{N-1} [q(v_k) - \lambda r(v_k)] \Big]. \end{aligned}$$

For each fixed λ, we have a problem which may be considered as above. The quantity λ is varied until the solution corresponding to the constraint (2.3b) is obtained.

5. DISCUSSION

The methods obtained above permit us to obtain a computational solution of the variational problem posed in §2.

The first member of the sequence, $\{f_K(c_1,c_2)\}$, is determined by the relation

$$(1) \qquad f_1(c_1,c_2) = \min_{v_0} \left[\delta\, h(c_1,c_2) + \delta\, g(v_0)\right],$$

and the succeeding members by means of (3.8). The computation of the maximum and the tabulation of the values is to be done by a digital computer.

As we might imagine, there is a bit more to the problem than what is apparently contained above. Leaving aside the important questions of grid size and stability, let us examine the memory problem. At each stage of the computation, the digital computer is engaged in computing the values of the function $f_K(c_1,c_2)$, based upon the values of $f_{K-1}(c_1,c_2)$. Consequently, it must store both sets of values in its memory, together with $v_0 = v(c_1,c_2)$.

Consider the case in which we take a range of 100 admissible values for c_1 and a like number for c_2. The range of pairs (c_1,c_2) will then encompass 10,000 values. This is a large number even by modern standards. Nevertheless, we have resolved problems of this type by these methods.

If, however, we consider problems involving two coupled systems, we meet four-dimensional systems, i.e. systems involving four state variables. The approach outlined above

would lead to a grid of 10^8 values.

Although there are a variety of techniques which can be used to reduce these formidable numbers, it is clear that if we wish to tackle really large scale systems, we must introduce some basic modifications of the method described above.

In the remainder of the paper we shall discuss some basic innovations.

6. PRELIMINARIES ON LINEAR DIFFERENTIAL EQUATIONS

In this section we collect some results concerning the solution of linear inhomogeneous equations which we shall require for our further discussion. Proofs of these results may be found in [9].

Let us consider first the case of second order equations, beginning with those with constant coefficients. If u is the solution of the equation

$$(1) \qquad u'' + a_1 u' + a_2 u = z(t), \quad u(0) = c_1, \quad u'(0) = c_2,$$

it may be written in the form

$$(2) \qquad u = u_0 + \int_0^t \left[\frac{e^{r_1(t-s)} - e^{r_2(t-s)}}{r_1 - r_2} \right] z(s)ds$$

where r_1 and r_2 are the roots of the characteristic equation

$$(3) \qquad r^2 + a_1 r + a_2 = 0,$$

assumed distinct for our purposes. The function u_0 is the solution of the homogeneous equation, having the form

$$(4)\qquad u = c_1u_1 + c_2u_2,$$

where u_1 and u_2 are the particular solutions of the homogeneous equation determined by the initial conditions

$$(5)\qquad \begin{aligned} &u_1(0) = 1, \quad u_2(0) = 0, \\ &u_1'(0) = 0, \quad u_2'(0) = 1. \end{aligned}$$

If a_1 and a_2 are functions of t, u possesses a representation of the form

$$(6)\qquad u = u_0 + \int_0^t k(t,s)z(s)ds$$

where

$$(7)\qquad k(s) = \frac{u_1(t)u_2(s) - u_2(t)u_1(s)}{u_1'(s)u_2(s) - u_1(s)u_2'(s)} .$$

The solutions u_1 and u_2 are determined by the boundary conditions of (5).

Corresponding results hold for the solutions of inhomogeneous linear differential equations of all order, and are most easily obtained via vector-matrix operations, cf. [9], Chapter 2.

7. A TERMINAL CONTROL PROBLEM

Let us now consider the following problem:

"Minimize

(1) $$|u(T)| + \lambda \int_0^T v^2 dt$$

over all $v(t)$ satisfying $m \leq v(t) \leq m'$, $0 \leq t \leq T$, where

(2) $$u'' + au' + bu = v(t), \quad u(0) = c_1, \quad u'(0) = c_2."$$

As we know from the previous sections, this problem can be treated computationally by means of sequences of functions of two variables. Let us now show that it can be reduced to a computational problem involving sequences of functions of one variable.

As in §6, the solution of (2) can be written

(3) $$u = u_0 + \int_0^t k(t-s)v(s)ds,$$

where $u_0 = c_1 u^{(1)} + c_2 u^{(2)}$, the solution of the homogeneous equation, and $k(t)$ is a linear combination, $b_1 e^{r_1 t} + b_2 e^{r_2 t}$, a known function.

The problem posed above can then be written:

"Minimize

(4) $$J(v) = \left| z + \int_0^T k(T-s)v(s)ds \right| + \lambda \int_0^T v^2 dt,$$

over all functions $v(t)$ satisfying $m \leq v(t) \leq m'$."

Let us introduce the function of z and T defined by the relation

$$(5) \qquad f(z,T) = \operatorname*{Min}_{v} J(v),$$

where $m \leq v(t) \leq m'$, $0 \leq t \leq T$.

To obtain a functional equation for $f(z,T)$, we argue as follows. Suppose that $v(s)$ has been chosen in an initial interval $[0,\delta]$. Then

$$(6) \qquad \begin{aligned} J(v) &= \left|z + \int_0^\delta k(T-s)v(s)ds + \int_\delta^T k(T-s)v(s)ds\right| \\ &\quad + \lambda\int_0^\delta v^2dt + \lambda\int_\delta^T v^2dt \\ &= \left|z + \int_0^\delta k(T-s)v(s)ds + \int_0^{T-\delta} k(T-\delta-s)v(s+\delta)ds\right| \\ &\quad + \lambda\int_0^\delta v^2dt + \lambda\int_0^{T-\delta} v^2(s+\delta)ds. \end{aligned}$$

It is clear that no matter how $v(s)$ has been chosen in $[0,\delta]$, we will choose $v(s)$ over $[\delta,T]$ so as to minimize the second expression in (6). This is a particular case of the principle of optimality, cf. [2]. Hence, for any choice of $v(s)$ in $[0,\delta]$, an optimal continuation yields

$$(7) \qquad J(v) = f\left(z + \int_0^\delta k(T-s)v(s)ds, T-\delta\right) + \lambda\int_0^\delta v^2dt.$$

Consequently, we choose $v(s)$ in $[0,\delta]$ so as to minimize this expression, obtaining the functional equation

$$(8) \qquad f(z,T) = \operatorname*{Min}_{v[0,\delta]} \left[f\left(z + \int_0^\delta k(T-s)v(s)ds, T-\delta\right) + \lambda\int_0^\delta v^2dt\right].$$

For computational purposes, we replace this by the approximate equation

$$(9) \qquad f(z,T) = \min_{v} \left[f(z+vk(T)\delta, T-\delta) + \lambda\delta v^2 \right],$$

where $m \leq v \leq m'$ and $T = 0, \delta, 2\delta, \ldots,$ with $f(z,0) = |z|$.

We have thus obtained the desired objective of replacing a numerical computation involving a sequence of functions of two-dimensional functions by one involving a sequence of one-dimensional functions.

8. A GENERAL CONTROL PROBLEM WITH LINEAR FEATURES

Let us now consider the following problem:

"Minimize

$$(1) \qquad J(v) = \int_0^T [g_1(t)u(t) + g_2(t)u'(t)]dt + b_1u(T) + b_2u'(T)$$

over all $v(t)$ satisfying the constraints

$$(2) \qquad \begin{aligned} &(a) \quad m \leq v(t) \leq m', \quad 0 \leq t \leq T, \\ &(b) \quad \int_0^T r(v(t))dt \leq k, \end{aligned}$$

and the relation

$$(3) \qquad u'' + a(t)u' + b(t)u = v, \quad u(0) = c_1, \quad u'(0) = c_2."$$

Let us first show that this problem is equivalent to that of minimizing a linear functional

$$(4) \qquad J_1(v) = \int_0^T h(t)v(t)dt$$

over all $v(t)$ satisfying the constraints of (2), where $h(t)$ is a known function.

To see this, we write $u(t)$ in the form

$$(5) \qquad u(t) = c_1u_1(t) + c_2u_2(t) + \int_0^t k(t,s)v(s)ds,$$

where, as we know from §6,

$$(6) \qquad k(t,s) = \frac{u_1(t)u_2(s) - u_2(t)u_1(s)}{w(s)} .$$

Then

$$(7) \qquad \begin{aligned} J(v) = c_3 + \int_0^T \Big[& g_1(t)\int_0^t k(t.s)v(s)ds \\ & + g_2(t)\int_0^t k_t(t,s)v(s)ds\Big]dt \\ & + b_1\int_0^T k(T,s)v(s)ds + b_2\int_0^T k_t(T,s)v(s)ds . \end{aligned}$$

The last two integrals have the desired form. Let us show that the first integral may also be written in this way. We have

$$
\begin{aligned}
(8)\qquad & \int_0^T g_1(t)\left(\int_0^t k(t,s)v(s)ds\right)dt \\
&= \int_0^T g_1(t)\left(\int_0^t \left[\frac{u_1(t)u_2(s) - u_2(t)u_1(s)}{w(s)}\right]v(s)ds\right)dt \\
&= \int_0^T \left[g_1(t)u_1(t)\int_0^t \frac{u_2(s)v(s)}{w(s)}ds\right. \\
&\quad \left. - g_1(t)u_2(t)\int_0^t \frac{u_1(s)v(s)}{w(s)}ds\right]dt.
\end{aligned}
$$

Integrating by parts, this becomes

$$
\begin{aligned}
&= \int_0^T \left[\frac{u_2(s)v(s)}{w(s)}\int_s^T g_1(t)u_1(t)dt\right. \\
&\quad \left. - \frac{u_1(s)v(s)}{w(s)}\int_s^T g_1(t)u_2(t)dt\right]ds.
\end{aligned}
$$

We see then that our problem reduces to maximizing $J_1(v)$, as given by (4), subject to the constraints of (2). This is a problem which in a number of cases can be solved analytically in explicit form cf. [13]. However, in order to indicate the methods we would employ in higher-dimensional cases where there were several v_i, let us discuss the functional equation approach.

Consider the function $f(k,z)$ defined by

$$
(9)\qquad f(k,z) = \underset{v}{\text{Min}} \int_z^T h(t)v(t)dt
$$

where v satisfies the constraints

$$(10)\quad \begin{aligned} &(a)\quad m \le v(t) \le m', \quad 0 \le t \le T, \\ &(b)\quad \int_z^T r(v(t))dt \le k. \end{aligned}$$

Then, as in §7, it is easy to obtain the functional equation

$$(11)\quad f(k,z) = \min_{m \le v \le m'} \left[\int_z^{z+\delta} h(t)v(t)dt + f\left(k - \int_z^{z+\delta} r(v(t))dt, z+\delta\right) \right].$$

For computational purposes, this reduces to

$$(12)\quad f(k,z) = \min_{m \le v \le m'} \left[h(z)v\delta + f(k - r(v)\delta, z+\delta) \right],$$

with $f(k,T) = 0$.

9. A NUMERICAL EXAMPLE

As an example of the foregoing, consider the problem of determining $v(t)$ so as to minimize

$$(1)\quad |u(10)| + \lambda \int_0^{10} v^2 dt,$$

where

$$(2)\quad u'' + .3u' + .02u = v, \quad u(0) = 2, \quad u'(0) = -.3.$$

The restrictions upon v are

$$(3)\quad -.1 \le v(t) \le 0.$$

Using the methods of §7, we convert the problem into that of computing the sequence $\{f_t(a)\}$ where

$$f_{10}(a) = |a|,$$

(4)

$$f_t(a) = \underset{v}{\text{Min}} \left[\lambda v^2 \Delta t + f_{t+\Delta t}(a+k(10-t)v\Delta t)\right].$$

The function $k(s)$ is defined by

(5) $$k(s) = 10(e^{-.1s} - e^{-.2s}).$$

The solution of the homogeneous linear equation $u'' + .3u' + .02u = 0$, with the assigned boundary conditions, is given by

(6) $$u(t) = e^{-.1t} + e^{-.2t},$$

so that $u(10) = e^{-1} + e^{-2} \cong .50$.

We wish to determine $f_0(.50)$.

For computing, we choose the parameters $\Delta t = .2$ sec, $\Delta a = .01$, which divides $0 \leq a \leq 50$ into 50 parts, $\lambda = 1.0$, and allow $v(t)$ to range between $-.1$ and 0. The values of $v(t)$ in this interval are taken at intervals of $.01$ between $-.10$ and $-.03$ and in intervals of $.003$ between $-.03$ and 0.

The optimal policy yields a value of $.0056$ for $\int_0^{10} v^2 dt$, and $u(10) = .0002$, so that $f_0(50) = .0058$. The optimal choice of $v(t)$, smoothed in order to compensate for our restriction to discrete values of v is shown below.

Four-point interpolation over a less dense a-grid yields the same results. The graph of $-v(t)$ is given below.

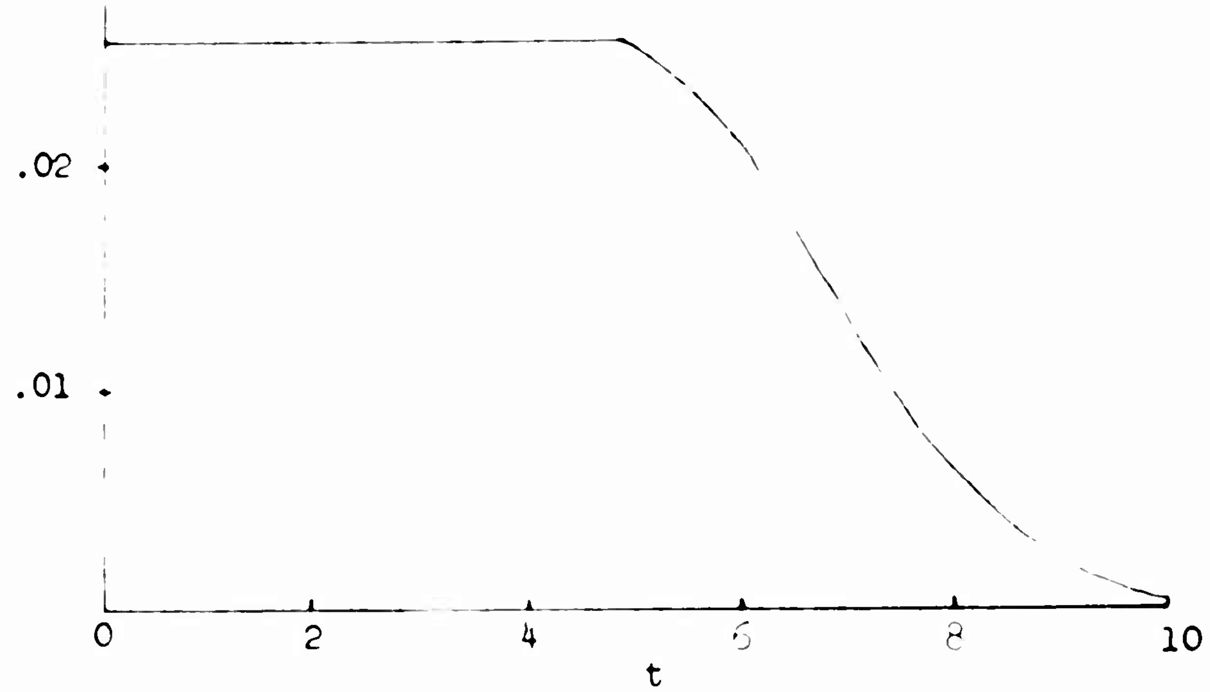

FIGURE 1

The answer required two hours on the RAND JOHNNIAC Computer, using a floating point interpretive scheme. The same calculations would consume fifteen minutes if an IBM-704 or comparable computer were used.

10. MULTI-DIMENSIONAL VERSION

Let us make two important observations.

I. The problem of minimizing a given function $h(x_1(T), x_2(T), \ldots, x_k(T))$ over all $v_i(t)$ satisfying the constraints

$$(1)\quad \begin{aligned} &(a)\quad m_i \le v_i(t) \le m_i', \quad i = 1, 2, \ldots, N, \quad 0 \le t \le T \\ &(b)\quad \int_0^T g(v_1, v_2, \ldots, v_N)\,dt \le b, \end{aligned}$$

and the relations

$$(2) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{N} a_{ij}(t)x_j + v_i, \quad i = 1, 2, \ldots, N,$$

$x_i(0) = c_i$, can always be treated by means of sequences of functions of k variables.

If N is large and $k = 1$ or 2, this means that we have a feasible method of solution using this new approach, where we could not treat the problem previously.

II. If the problem is that of minimizing

$$(3) \qquad J(v) = \int_0^T \left[\sum_{i=1}^{N} h_i(t)x_i(t) \right] dt + \sum_{i=1}^{N} g_i x_i(T),$$

the problem may be treated by means of sequences of functions of one variable, regardless of the size of N.

11. USE OF SUCCESSIVE APPROXIMATIONS

We have seen in the previous sections that the problem of minimizing a functional of the form

$$(1) \qquad J(v) = \int_0^T \left[\sum_{i=1}^{N} h_i(t)x_i(t) \right] dt + \sum_{i=1}^{N} b_i x_i(T)$$

over all functions $v_i(t)$ satisfying the constraints

$$(2) \qquad \begin{aligned} &(a) \quad m_i \leq v_i(t) \leq m_i', \quad i = 1, 2, \ldots, N, \quad 0 \leq t \leq T, \\ &(b) \quad \int_0^T g(v_1, v_2, \ldots, v_N)dt \leq k, \end{aligned}$$

where the x_i are determined by linear homogeneous equations of the form

$$
(3) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{N} a_{ij}(t)x_j + v_i(t), \quad i = 1, 2, \ldots, N,
$$
$$
x_i(0) = c_i,
$$

can be resolved numerically by means of sequences of functions of one variable, regardless of the magnitude of N.

Essential in this reduction was the superposition property of linear systems. Let us now indicate how the method of successive approximations may be applied to yield an approach to nonlinear systems by means of these methods.

Consider the problem of minimizing a functional of the form

$$
(4) \qquad J(v) = \int_0^T F(x_1, x_2, \ldots, x_N)dt + G(x_1(T), x_2(T), \ldots, x_N(T))
$$

over all v_i satisfying (2), where

$$
(5) \qquad \frac{dx_i}{dt} = H_i(x_1, x_2, \ldots, x_N) + v_i(t), \quad i = 1, 2, \ldots, N,
$$
$$
x_i(0) = c_i.
$$

Let $v_i^{(0)}(t)$, $i = 1, 2, \ldots, N$, $0 \leq t \leq T$, be an initial choice of the $v_i(t)$, and $x_i^{(0)}(t)$, $i = 1, 2, \ldots, N$, the values of x_i determined by (5). Consider the new problem of minimizing the linear functional

$$J_L(v) = \int_0^T \Big[F(x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}) + \sum_{i=1}^{N} (x_i - x_i^{(0)}) \frac{\partial F}{\partial x_i^{(0)}} \Big] dt \tag{6}$$

$$+ G(x_1^{(0)}(T), x_2^{(0)}(T), \ldots, x_N^{(0)}(T))$$

$$+ \sum_{i=1}^{N} (x_i(T) - x_i^{(0)}(T)) \frac{\partial G}{\partial x_i^{(0)}}$$

subject to the linear equations

$$\frac{dx_i}{dt} = H_i(x_1^{(0)}, x_2^{(0)}, \ldots, x_N^{(0)}) + \sum_{i=1}^{N} (x_i - x_i^{(0)}) \frac{\partial H}{\partial x_i^{(0)}} + v_i(t), \tag{7}$$

$x_i(0) = c_i$, $i = 1, 2, \ldots, N$. The functions $v_i(t)$, as before, satisfy (2).

As a linear problem, the minimizing $v_i(t)$ can be determined by means of sequences of functions of one variable. The numerical solution of the foregoing problem is thus a relatively simple matter.

Having determined the new functions $v_i^{(1)}$ and $x_i^{(1)}$ in this fashion, we replace $v_i^{(0)}$ and $x_i^{(0)}$ by $v_i^{(1)}$ and $x_i^{(1)}$ and continue the process.

The original nonlinear variational problem has thus been replaced by a sequence of linear variational problems. Consequently, the original nonlinear variational problem,

requiring a sequence of functions of N variables for its solution, has been replaced by a sequence of problems requiring functions of one variable.

The question of the convergence of this sequence of solutions to the solution of the original problem is one which requires a detailed treatment which we shall not present here.

12. TIME-LAG PROBLEMS

If the underlying equation of the system has the form

$$(1) \qquad \frac{d^2u}{dt^2} = g\left(\frac{du}{dt}, u(t), u(t-t_1), \ldots, u(t-t_k)\right),$$

corresponding to time-lags and retarded control, the direct method of dynamic programming presented in §3 yields no feasible computational scheme.

If the equation is linear, and the criterion function is linear, representation theorems akin to those given in §5, cf. [10], [11], permit us to obtain results corresponding to those obtained above.

These and further results pertaining to control problems for more general types of functional equations will be presented in detail in other publications.

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DIRECTOR 1:	Roger, White ... understand.
	(TURNS TO SENIOR DIRECTOR)
	White has a B-29-type Bogie that's track Charlie 29, no marking.
SENIOR DIR:	No marking? One aircraft?
DIRECTOR 1:	One aircraft.
MI OFFICER:	Charlie 29?
DIRECTOR 1:	Charlie 29.
SENIOR DIR:	Nothing on 29? (TO MI OFFICER)
CUT TO:	(CPB SHOT)
NARRATOR:	Good decisions are not necessarily all fast. Because Charlie 29 is farther away, the Senior Director tries to get more information before opening fire.
DIRECTOR 1:	(TO MI OFFICER)
	30 has two ... Go ahead, Red! Roger.
CUT BACK TO:	(FULL STATION SHOT)
SENIOR DIR:	(TURNS TO MI OFFICER)
	What did you get on 29?
MI OFFICER:	We have nothing, sir.
SENIOR DIR:	POUNCE 29.
DIRECTOR 1:	(IN BACKGROUND)
	Roger, Red. Maintain surveillance.
CONTROL TECH:	Kingpin, we ordered a splash on 29.
DIRECTOR 1:	Redbird White, POUNCE. Over.
CUT TO:	(REAR VIEW OF TOP DECK WITH FOUR EXPERIMENTERS IN FOREGROUND)

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